Minimization with constraint and interpretation of lambda

A company wants to minimize its total costs, with the condition that the revenue obtained from selling quantities x_1, x_2 of the products it manufactures exceeds a certain minimum threshold. If the unit costs of manufacturing each product are linear functions of the outputs produced in the form $C_1 = x_1$ and $C_2 = 2x_2$ and the selling prices of the products are $p_1 = 1$ and $p_2 = 3$ respectively, it is required:

1. Solve the problem assuming they must earn at least 3 monetary units.
2. Study how the minimum cost will vary if they earn 2.8 monetary units. And if they earn 3.1?

Solution

1. The constraint is:

$$x_1 + 3x_2 = 3$$

We set up the Lagrangian for the problem:

$$L = x_1^2 + 2x_2^2 + \lambda(3 - x_1 - 3x_2)$$

We calculate the first-order conditions:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 4x_2 - 3\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 3 - x_1 - 3x_2 = 0$$

From the first equation we get:

$$\lambda = 2x_1$$

From the second equation we get:

$$\lambda = \frac{4x_2}{3}$$

Equating both expressions for λ:

$$2x_1 = \frac{4x_2}{3}$$

Solving for one of the variables:

$$x_2 = \frac{3}{2}x_1$$

Substitute into the third condition:

$$3 - x_1 - 3\left(\frac{3}{2}x_1\right) = 0$$

$$3 - x_1 - \frac{9}{2}x_1 = 0$$

$$3 - \frac{11}{2}x_1 = 0$$

$$3 = \frac{11}{2}x_1$$

$$x_1 = \frac{6}{11}$$

With this we obtain the value of x_2:

$$x_2 = \frac{3}{2} \cdot \frac{6}{11} = \frac{18}{22} = \frac{9}{11}$$

The minimum cost value is obtained by inserting x_1 and x_2 into the cost function:

$$C = x_1^2 + 2x_2^2$$

$$C = \left(\frac{6}{11}\right)^2 + 2\left(\frac{9}{11}\right)^2$$

$$C = \frac{198}{121} \approx 1.636$$

Therefore, the minimum cost is approximately 1.636 monetary units.

2. To determine how the minimum cost will change, we calculate the value of λ:

$$\lambda^* = 2 \cdot \frac{6}{11} = \frac{12}{11}$$

And now we multiply the change in income by the value of λ:

$$(2.8 - 3) \cdot \frac{12}{11} = -0.218$$

$$(3.1 - 3) \cdot \frac{12}{11} = 0.109$$